Boson-Fermion Confusion: The String Path To Supersymmetry

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Reminiscences on the String origins of Supersymmetry are followed by a discussion of the importance of confusing bosons with fermions in building superstring theories in $9+1$ dimensions. In eleven dimensions, the kinship between bosons and fermions is more subtle, and may involve the exceptional group F_4.

1. Introduction

Although the idea of a symmetry between bosons and fermions must be very old, after all both are present in Nature, I am only aware of non-relativistic attempts in that direction in the 1960's. Stavraki [1] (1967) proposed a current algebra with both commutators and anticommutators. Myazawa [2] (1968), then at the University of Chicago, put together the fermions in the **56** with the bosons in the **35** of $SU(6)$ into one algebraic structure, which led him to invent the superalgebra we now call $SU(6/21)$!

In the Soviet Union, a brilliant generalization of the Poincaré group to include anticommuting charges is proposed in 1971 by Golfand and Likhtman [3], realizing relativistic supersymmetry in $3+1$ dimensions for the first time. This is followed in 1972 by a non-linear realization of this symmetry by Volkov and Akulov [4].

String Theory stems from the Dual Resonance Model, formulated to satisfy Dolen-Horn-Schmid duality [5], according to which, in pion-nucleon scattering, the averaged s-channel fermionic resonances were related to the t-channel boson exchanges. As such it implied a relation between bosons and fermions, although early workers seem to have put aside spin as an inessential complication. Amplitudes involving only bosons, known today as the Veneziano (open string) and Virasoro-Shapiro (closed string) models, are the progenitors of modern string theories. The generalization of Dual Models to include half-odd integer spins [6,7] followed soon after in early 1971. When expressed in terms of world-sheet symmetries, both R and NS formulations were shown [8] to be examples of supersymmetry on the $1+1$ world-sheet. This in turn led to supersymmetry in $3+1$ dimensions, with explicit local interacting field-theories [9]. It is only later that space-time supersymmetry between the R and NS sectors was realized in $9+1$ dimensions by the GSO [10] projection. Born in the context of Dual Resonance Models, relativistic supersymmetry has now been found to play a central role in the formulation of quantum field theories, and perhaps even of Nature itself.

2. Reminiscences

This conference affected me like a *madeleine*, and although still quite young, I will take a few lines to offer some recollections of the epic period when the building blocks of supersymmetry were being laid down. Soon after graduating from Syracuse University in spring 1969, my wife and I sailed to Europe to spend the summer at the ICTP, where J. Nuyts and H. Sugawara introduced me to the joys of working on the Veneziano model. By the time we sailed back to the United States to join the Fermi National Laboratory (then known as NAL) as one of its first postdocs, I had been hopelessly seduced by their elegance and promise of simplicity. So much

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so that, in the middle of the Atlantic, I found myself in the reading room of the liner *France*, with a seminal paper by Fubini and Veneziano on the harmonic oscillator formulation. I then went to my cabin to fetch postcards. Upon my return I saw the Fubini-Veneziano preprint on the table I had vacated. Doubly puzzled, since I had just left it in my cabin, and it was, unlike my copy, heavily annotated, I decided to wait for its owner. In walks a rather lanky individual to reclaim his posession; it was André Neveu, on his way to join Joël Scherk in Princeton!

In the Fall of 1969, at NAL, I started working with Lou Clavelli on the group theoretical structure of the Veneziano amplitudes. There were no senior theorists at NAL; we were on our own, but for Nambu, who was very supportive of our work, and even invited us to lunch at the Quadrangle Club! I also met in that period, many of the early luminaries: M. Virasoro, then at Madison, Scherk and his first wife (whom he had met at the Club Méditerranée!), S. Fubini, B. Sakita, and G. Veneziano.

In the spring of 1970, the Director of NAL, Bob Wilson decreed: "All theorists must go to Aspen". I had never heard of the place, did not want to go, but one did not resist Bob very long. That summer, at the Aspen Center for Physics, the break from grungy calculations, combined with the mountain air and the wonderful music, made me realize an easy way to look at these models. I decided to use the generalized position and momentum operators, introduced by Fubini and Veneziano, in devising equations of motion. Upon my return at NAL, I wrote a short paper on the Dual Klein-Gordon equation, where I spoke of the correspondance between Dual systems and point particles (later to be known as the zero-slope limit). To my dismay, it was rejected by Physics Letters, and I withdrew the paper. Shaken but undaunted, I proceeded to generalize the Dirac equation. There I found to my surprise a new algebraic structure, the square root of the Virasoro algebra. That Fall, I went to the Institute in Princeton where Nambu was spending a sabbatical. Again he was very supportive of my ideas, and encouraged me further, but I got stuck because I did not know Grassmann variables! It is during that visit that I met Mike Green, but I had to wait until the spring of 1971 to meet John Schwarz, and even longer L. Brink, and C. Thorn, my lifelong dual friends!

3. The First Fermion-Boson Confusion

Fermions and bosons cannot easily be confused, as they differ both by their quantization and space-transformation properties, but the latter can be very similar in some dimensions. An early example, where assigning bosonic quantum numbers to fermions leads to a relativistic theory, is the introduction of fermions to Dual Models [6]. The N-point (bosonic) Veneziano amplitude is a sum of terms of the form

$$< 1\,|V(2)V(3)\cdots V(N-1)|\,N\, > \,, \tag{1}$$

where

$$V(j) \sim \;\; : e^{ik_j\cdot Q(\theta_j)} : \,, \tag{2}$$

is the vertex for the emission of the jth particle of momentum k_j, which resembles a plane wave with a generalized coordinate

$$Q_\mu(\theta) = x_\mu + \theta p_\mu + \sum_1^\infty \frac{1}{\sqrt{n}}\Big(a_\mu^{(n)\dagger}e^{-in\theta} + a_\mu^{(n)}e^{in\theta}\Big)\,. \tag{3}$$

Here x_μ and p_μ are the usual position and momentum, and there are an infinite number of relativistic harmonic oscillators

$$[\,a_\mu^{(n)}, a_\rho^{(m)\dagger}\,] = \delta^{nm} g_{\mu\rho}\ . \tag{4}$$

This leads to a generalized momentum

$$P_\mu(\theta) = \frac{dQ_\mu(\theta)}{d\theta} \;=\; p_\mu - i\sum_1^\infty \sqrt{n}\Big(a_\mu^{(n)\dagger}e^{-in\theta} - a_\mu^{(n)}e^{in\theta}\Big)\,. \tag{5}$$

Both operators reduce to their point limit in the (zero slope) limit

$$Q_\mu(\theta) \;\;\to x_\mu\ ; \qquad P_\mu(\theta) \;\;\to p_\mu\ . \tag{6}$$

Now if we apply this correspondance to the Klein-Gordon operator,

$$p_\mu\, p^\mu\, \Phi = 0 \quad \to P_\mu(\theta)\, P^\mu(\theta)\, \Phi = 0\ , \tag{7}$$

with

$$P_\mu(\theta)\, P^\mu(\theta) = L_0 + \sum_1^\infty \Big(L_n e^{-in\theta} + L_{-n}e^{in\theta}\Big)\,, \tag{8}$$

where the L's satisfy the Virasoro algebra. In particular, L_0 is, up to an additive constant, the equation of motion for the bosons, and the $L_n \sim p \cdot a^{(n)}$ are akin to the decoupling operators of QED. Hence the analogy is established: take the generalized quantities, and replace the product of the averages by the average of the products:

$$< P_\mu(\theta) >< P^\mu(\theta) > \quad \rightarrow \quad < P_\mu(\theta)\, P^\mu(\theta) > , \tag{9}$$

where $< \ldots >$ denotes the average, or integration over θ.

The same procedure can be applied to the Dirac equation

$$\gamma \cdot p\, \Psi \;=\; 0 \;, \tag{10}$$

by imagining that the gamma matrices are themselves averages of something

$$\gamma_\mu \quad \rightarrow \quad \Gamma_\mu(\theta) \;=\; \gamma_\mu + \cdots \;, \tag{11}$$

leading to a generalized Dirac equation

$$< \Gamma(\theta) \cdot P(\theta) > \Psi \;=\; 0 \;, \tag{12}$$

together with

$$\{\, \Gamma_\mu(\theta), \Gamma_\rho(\theta') \,\} \;=\; 2 g_{\mu\rho} \delta(\theta - \theta') \;. \tag{13}$$

this requires the introduction of anticommuting harmonic oscillators which carry *vector* indices

$$\Gamma_\mu(\theta) = \gamma_\mu + \gamma_5 \sum_1^\infty \left(b_\mu^{(n)\dagger} e^{-in\theta} + b_\mu^{(n)} e^{in\theta} \right) , \tag{14}$$

where γ_5 anticommutes with the Dirac matrices. Then

$$\Gamma(\theta) \cdot P(\theta) = F_0 + \sum_1^\infty \left(F_n e^{in\theta} + F_{-n} e^{-in\theta} \right) , \tag{15}$$

where the F's satisfy the superVirasoro algebra. This chain of reasoning covers the genesis of the Dirac equation and the appearance of commuting and anticommuting structures in a relativistic framework. It is merely a generalization of the algebra of Dirac's operator

$$\{\, \gamma \cdot p \,,\, \gamma \cdot p \,\} \;=\; 2\, p \cdot p \;. \tag{16}$$

This procedure, assigning a vector index to a fermion, might appear foolish at first glance, except in 9+1 dimensions where the light-cone little group is $SO(8)$, spinors and vectors have the same number of degrees of freedom.

In relativistic theories, bosons and fermions usually transform differently under space rotations. For massless particles the relevant group of rotations is the light-cone little group. In $1+1$ dimensions, there is no such group, and bosons differ from fermions only by quantization, and one can build bosons out of fermions without group-theoretical obstructions; the same applies to $2+1$ dimensions. In $3+1$ dimensions, the little group is non-trivial and fermions are distinguished by their helicities– integer for bosons, half-integer for fermions. In higher dimensions, fermions (bosons) transform according to the spinor (tensor) representations of the Non-Abelian little group. In $9+1$ dimensions, the massless little group is $SO(8)$ and bosons and fermions have the same number of degrees of freedom. This fact lies at the heart of the superstring constructions. In $10+1$ dimensions and above, they become different again. However, in special numbers of dimensions, a strange kinship between spinor and tensor representations of the appropriate rotation group appears. In eleven dimensions, it leads to the supergravity theory, and, as we will show, possibly more.

4. A Second Fermion-Boson Confusion?

In $10+1$ dimensions, there is no apparent kinship between fermions and bosons. Yet there exists a supersymmetric theory in eleven dimensions, M-theory, with supergravity as its local limit. The degrees of freedom of supergravity are massless particles, belonging to representations of $SO(9)$, the light-cone little group:

- Graviton as a symmetric second-rank tensor $h_{(ij)}$, with Dynkin label [2000],
- Third-rank antisymmetric tensor, $A_{[ijk]}$, with Dynkin label [0010],
- Rarita-Schwinger spinor-vector, $\Psi_{\alpha i}$, with label [1001].

Their group-theoretical properties are summarized in the following table of Dynkin indices of different orders [11]

irrep	[1001]	[2000]	[0010]
I_0	128	44	84
I_2	256	88	168
I_4	640	232	408
I_6	1792	712	1080
I_8	5248	2440	3000

These indices, except for I_8, match between the fermion and the two bosons. It turns out that there are infinitely many trios of representations of $SO(9)$ with similar group-theoretic relations among them. The simplest example is given by the triplet made of fields with index structure

$$h_{(ijk)l} + A_{(ij)(kl)m} + \Psi_{\alpha(ij)k} \ , \tag{17}$$

and group-theoretical properties

irrep	[2100]	[0110]	[1101]
I_0	910	1650	2560
I_2	3640	6600	10240
I_4	19864	34920	54784
I_6	130840	217320	348160
I_8	977944	1498344	2466304

They describe higher spin massless fields, with no apparent supersymmetry. This is only one example of this infinite set, which can be obtained from a character formula[12], traced to the three equivalent embeddings of $SO(9)$ inside the exceptional group F_4! Under the embedding $F_4 \supset SO(9)$, the 52 parameters of F_4 contain the 36 generators of $SO(9)$ and 16 parameters which transform as the $SO(9)$ spinor representation, and label the coset F_4/SO_9. Kostant[13] introduces over that space sixteen (256×256) gamma matrices which generate the Clifford algebra

$$\{\,\gamma^a\,,\,\gamma^b\,\} \;=\; 2\,\delta^{ab}\ , \quad a,b = 1,2\ldots,16\ . \tag{18}$$

Note that the "vector indices" of these matrices actually transform as the spinor of $SO(9)$! This is possible because of the anomalous embedding $SO(16)\supset SO(9)$, where the 16 vector of $SO(16)$ is the 16 spinor of $SO9)$. Another example of fermion-boson confusion. Let T^a be the generators of F_4 not contained in $SO(9)$, and form the Kostant equation

$$\sum_1^{16}\gamma^a\,T^a\,\Psi \;\equiv\; \not{K}\,\Psi = \;0\ . \tag{19}$$

Its solutions consist of all triples, including the supergravity multiplet. It is convenient to rewrite the gamma matrices in terms of eight Grassmann variables, and express the solutions as chiral superfields in these variables,

$$\Psi \;=\; \psi_0 + \psi_i\theta_1 + \psi_{ij}\theta_i\theta_j + \cdots\ , \tag{20}$$

and the supergravity solution corresponds to all constant ψ. Under $SO(9)\supset SO(7)\times SO(2)$, these split as

$$\mathbf{1}+\mathbf{8}+\mathbf{28}+\mathbf{56}+\mathbf{70}+\mathbf{56}+\mathbf{28}+\overline{\mathbf{8}}+\mathbf{1}\ , \tag{21}$$

reproducing the supergravity multiplet. Other solutions involve fields of higher spin. If these fields are to be incorporated in a relativistic theory, we must overcome the problem of massless spins with spins higher than two, bringing in well documented difficulties, with coupling spin-one current [14] and energy-momentum tensor [15,16] to massless particles of spin greater than one.

5. A Simpler Example

A similar but much simpler construction can be achieved for the coset $SU(3)/SU(2)\times U(1)$. At the lowest level, it leads to a triplet of representations on which $N=2$ supersymmetry can be realized in $3+1$ dimensions.

5.1. The $N=2$ Hypermultiplet

We first recall the well-known light-cone description of the massless $N=2$ hypermultiplet in $3+1$ dimensions [17], which contains two Weyl spinors and two complex scalar fields, on which the $N=2$ SuperPoincaré algebra is realized. Introduce the light-cone Hamiltonian

$$P^- = \frac{p\overline{p}}{2p^+}\ , \tag{22}$$

where $p = \frac{1}{\sqrt{2}}(p^1+ip^2)$. The front-form supersymmetry generators satisfy the anticommutation relations

$$\begin{aligned}
\{\mathcal{Q}^m_+,\overline{\mathcal{Q}}^n_+\} &= -2\delta^{mn}p^+\ ,\\
\{\mathcal{Q}^m_-,\overline{\mathcal{Q}}^n_-\} &= -2\delta^{mn}\frac{p\overline{p}}{p^+}\ , \qquad m,n=1,2\ ,\\
\{\mathcal{Q}^m_+,\overline{\mathcal{Q}}^n_-\} &= -2p\delta^{mn}\ .
\end{aligned} \tag{23}$$

The kinematic supersymmetries are expressed as

$$\mathcal{Q}^m_+ = -\frac{\partial}{\partial\overline{\theta}^m} - \theta_m p^+ \ , \quad \overline{\mathcal{Q}}^m_+ = \frac{\partial}{\partial\theta^m} + \overline{\theta}_m p^+ \ , \tag{24}$$

while the kinematic Lorentz generators are given by

$$\begin{aligned} M^{12} &= i(x\overline{p} - \overline{x}p) + \frac{1}{2}\theta_m \frac{\partial}{\partial\theta_m} - \frac{1}{2}\overline{\theta}^m \frac{\partial}{\partial\overline{\theta}^m} \ , \\ M^{+-} &= -x^- p^+ - \frac{i}{2}\theta_m \frac{\partial}{\partial\theta_m} - \frac{i}{2}\overline{\theta}^m \frac{\partial}{\partial\overline{\theta}^m} \ , \\ M^+ &\equiv \frac{1}{\sqrt{2}}(M^{+1} + iM^{+2}) = -xp^+ \ , \\ \overline{M}^+ &= -\overline{x}p^+ \ , \end{aligned} \tag{25}$$

where the two complex Grassmann variables satisfy the anticommutation relations

$$\begin{aligned} \{\theta_m, \frac{\partial}{\partial\theta_n}\} &= \{\overline{\theta}^m, \frac{\partial}{\partial\overline{\theta}^n}\} = \delta^{mn} \ , \\ \{\theta_m, \frac{\partial}{\partial\overline{\theta}^n}\} &= \{\overline{\theta}^m, \frac{\partial}{\partial\theta_n}\} = 0 \ . \end{aligned}$$

The (free) Hamiltonian-like supersymmetry generators are simply

$$\mathcal{Q}^m_- = \frac{\overline{p}}{p^+}\mathcal{Q}^m_+ \ , \qquad \overline{\mathcal{Q}}^m_- = \frac{p}{p^+}\overline{\mathcal{Q}}^m_+ \ , \tag{26}$$

and the light-cone boosts are given by

$$\begin{aligned} M^- &= x^- p - \frac{1}{2}\{x, P^-\} + i\frac{p}{p^+}\theta_m \frac{\partial}{\partial\theta_m} \ , \\ \overline{M}^- &= x^- \overline{p} - \frac{1}{2}\{\overline{x}, P^-\} + i\frac{\overline{p}}{p^+}\overline{\theta}^m \frac{\partial}{\partial\overline{\theta}^m} \ , \end{aligned} \tag{27}$$

where

$$x = \frac{1}{\sqrt{2}}(x^1 + ix^2) \ .$$

These generators represent the superPoincaré algebra on reducible superfields because the operators

$$\mathcal{D}^m_+ = \frac{\partial}{\partial\overline{\theta}^m} - \theta_m p^+ \ , \tag{28}$$

anticommute with the supersymmetry generators. Irreducibility is achieved by acting on superfields for which

$$\mathcal{D}^m_+ \ \Phi = [\frac{\partial}{\partial\overline{\theta}^m} - \theta_m p^+]\Phi = 0 \ , \tag{29}$$

solved by the chiral superfield

$$\Phi(y^-, x^i, \theta_m) = \psi_0 + \theta_m \psi^m + \theta_1\theta_2\psi^{12} \ , \tag{30}$$

where the arguments of the ψ's depend on

$$y^- = x^- - i\theta_m\overline{\theta}^m \ , \tag{31}$$

and the transverse variables. Acting on this chiral superfield, the constraint is equivalent to requiring that

$$\mathcal{Q}^m_+ \approx -2p^+\theta_m \ , \qquad \overline{\mathcal{Q}}^m_+ \approx \frac{\partial}{\partial\theta_m} \ , \tag{32}$$

where the derivative is meant to act only on the naked θ_m's, not on those hiding in y^-.

5.2. Coset Construction

The degrees of freedom of the $N = 2$ hypermultiplet in four dimensions appear as trivial solutions of the Kostant equation associated with the coset $SU(3)/SU(2) \times U(1)$. Let T^A , $A = 1, 2, \ldots, 8$, be the generators of $SU(3)$. Among those, T^i, $i = 1, 2, 3$, and T^8 generate its $SU(2)\times U(1)$ subalgebra. Introduce Dirac matrices over the coset

$$\{\gamma^a, \gamma^b\} = 2\delta^{ab} \ , \quad a, b = 4, 5, 6, 7 \ . \tag{33}$$

The Kostant equation over the coset $SU(3)/SU(2) \times U(1)$

$$\not{K}\,\Psi = \sum_{a=4,5,6,7} \gamma^a T_a \Psi = 0 \ , \tag{34}$$

has an infinite number of solutions which come in groups of three representations of $SU(2) \times U(1)$, called Euler triplets. For each representation of $SU(3)$, there is a unique Euler triplet, $\{a_1, a_2\}$:

$$[a_2]_{-\frac{2a_1+a_2+3}{6}} \oplus [a_1{+}a_2{+}1]_{\frac{a_1-a_2}{6}} \oplus [a_1]_{\frac{2a_2+a_1+3}{6}} \ , \tag{35}$$

where a_1, a_2 are the Dynkin labels of the associated $SU(3)$ representation. Here, $[a]$ stands for the $a = 2j$ representation of $SU(2)$, and the subscript denotes the $U(1)$ charge. Kostant's operator commutes with the $SU(2) \times U(1)$ generated by

$$L_i = T_i + S_i \ , \ i = 1, 2, 3 \ ; \qquad L_8 = T_8 + S_8 \ , \tag{36}$$

a sum of the $SU(3)$ generators and the "spin" part, expressed in terms of the γ-matrices as

$$S_j = -\frac{i}{4} f_{jab}\gamma^{ab} \ , \qquad S_8 = -\frac{i}{4} f_{8ab}\gamma^{ab} \ , \tag{37}$$

where $\gamma^{ab} = \gamma^a\gamma^b$, $a \neq b$, and f_{jab} , f_{8ab} are structure functions of $SU(3)$. The Euler triplet corresponding to $a_1 = a_2 = 0$,

$$\{0,0\} \ = \ [0]_{-\frac{1}{2}} \oplus [1]_0 \oplus [0]_{\frac{1}{2}} \ , \tag{38}$$

describes the degrees of freedom of the $N = 2$ supersymmetric multiplet, when the $U(1)$ is interpreted as the helicity of the four-dimensional Poincaré algebra.

Is it possible to link this supersymmetric triplet to the others for which $a_{1,2} \neq 0$, while preserving relativistic invariance? Not all triplets can decribe relativistic particles, since their $U(1)$ charges are in general fractional numbers, leading to states that pick up strange phases after a space rotation by 2π, while Fermi-Dirac statistics only allows states for which this phase is ± 1. Only Euler triplets for which

$$a_1 - a_2 = 3n \ , \tag{39}$$

where $n = 0, \pm 1, \pm 2, \ldots$, yield half-odd integer or integer $U(1)$ charges fit the bill. These Euler multiplets split into two groups, the self-conjugate,

$$\{a,a\} \ : \qquad [a]_{-\frac{a+1}{2}} \oplus [2a+1]_0 \oplus [a]_{\frac{a+1}{2}} \ , \tag{40}$$

which contain equal number of half-odd integer-helicity fermions and integer-helicity bosons, and naturally satisfy CPT. The other possible Euler multiplets are of the form $\{a, a+3n\}$ with $n = 1, 2, \ldots$,

$$[a]_{-\frac{a+2n+1}{2}} \oplus [2a+3n+1]_{\frac{n}{2}} \oplus [a+3n]_{\frac{a+n+1}{2}} \ .$$

Since CPT requires states of opposite helicity, these must be accompanied by their conjugates, $\{a+3n, a\}$, with all helicities reversed. If both n and a are even, each representation contains $(2a+3n+2)$ bosons and fermions, the fermions appearing in two different $SU(2)$ representations.

The helicities within each triplet are separated by more than half a unit, and they cannot be related by operations, such as supersymmetry, which change helicity by half a unit. Thus a necessary condition for supersymmetry to be realized is to include all triplets, leading to an infinite-component theory.

We also note that there are states in the higher Euler triplets with helicities larger than 2. If they are to be interpreted as massive relativistic states, they must arrange themselves in $SO(3)$ representations, which does not appear likely. Otherwise they must be interpreted as massless particles in four dimensions, lealing with a theory of massless states of spin higher than two.

There are well-known difficulties with such theories[15,16]. In particular, they do not have covariant energy momentum tensors, and it must be that in the flat space limit they decouple from the gravitational sector. Alternatively, the no-go theorems do not apply if there are an infinite number of such particles. The best argument against such theories is that no working example has yet been produced, but we hope such a theory can be formulated with an infinite number of Euler multiplets.

5.3. Grassmann Numbers and Dirac Matrices

In order to make contact with the supersymmetry of the lowest Euler triplet, we represent [18] the γ-matrices in terms of Grassmann numbers and their derivatives as

$$\begin{aligned} \gamma_{4+i5} &= i\sqrt{\frac{2}{p^+}}\,\mathcal{Q}^1_+ \ , \quad \gamma_{4-i5} = i\sqrt{\frac{2}{p^+}}\,\overline{\mathcal{Q}}^1_+ \\ \gamma_{6+i7} &= i\sqrt{\frac{2}{p^+}}\,\mathcal{Q}^2_+ \ , \quad \gamma_{6-i7} = i\sqrt{\frac{2}{p^+}}\,\overline{\mathcal{Q}}^2_+ \ , \end{aligned}$$

in terms of the kinematic $N = 2$ light-cone supersymmetry generators defined in the previous section. It follows that the Kostant operator anticommutes with the constraint operators

$$\{ \ \not{K}, \ \mathcal{D}^m_+\} \ = \ 0 \ , \tag{41}$$

so that we can simplify its solutions to chiral superfields, on which these become

$$\begin{aligned} \gamma_{4+i5} &= -2i\sqrt{2p^+}\ \theta_1 \ , \quad \gamma_{4-i5} = i\sqrt{\frac{2}{p^+}}\ \frac{\partial}{\partial\theta_1} \\ \gamma_{6+i7} &= -2i\sqrt{2p^+}\ \theta_2 \ , \quad \gamma_{6-i7} = i\sqrt{\frac{2}{p^+}}\ \frac{\partial}{\partial\theta_2} \ , \end{aligned}$$

The "spin" parts of the $SU(2)\times U(1)$ generators, expressed in terms of Grassmann variables, do not depend on p^+,

$$\begin{aligned}
S_1 &= \frac{1}{2}(\theta_1\frac{\partial}{\partial\theta_2}+\theta_2\frac{\partial}{\partial\theta_1})\ ,\\
S_2 &= -\frac{i}{2}(\theta_1\frac{\partial}{\partial\theta_2}-\theta_2\frac{\partial}{\partial\theta_1})\ ,\\
S_3 &= \frac{1}{2}(\theta_1\frac{\partial}{\partial\theta_1}-\theta_2\frac{\partial}{\partial\theta_2})\ ,
\end{aligned}$$

and

$$S_8=\frac{\sqrt{3}}{2}(\theta_1\frac{\partial}{\partial\theta_1}+\theta_2\frac{\partial}{\partial\theta_2}-1)\ ,$$

identified with the helicity, up to a factor of $\sqrt{3}$.

5.4. Linear Realization of $SU(3)$

The $SU(3)$ generators can be conveniently expressed on three complex variables and their conjugates. Define for convenience the differential operators

$$\partial_1\equiv\frac{\partial}{\partial z_1}\ ,\qquad \overline{\partial}_1\equiv\frac{\partial}{\partial\overline{z}_1}\ ,\ \text{etc.}\ ,$$

in terms of which the generators are given by

$$\begin{aligned}
T_1+iT_2 &= z_1\partial_2-\overline{z}_2\overline{\partial}_1\ , & T_1-iT_2 &= z_2\partial_1-\overline{z}_1\overline{\partial}_2\ ,\\
T_4+iT_5 &= z_1\partial_3-\overline{z}_3\overline{\partial}_1\ , & T_4-iT_5 &= z_3\partial_1-\overline{z}_1\overline{\partial}_3\ ,\\
T_6+iT_7 &= z_2\partial_3-\overline{z}_3\overline{\partial}_2\ , & T_6-iT_7 &= z_3\partial_2-\overline{z}_2\overline{\partial}_3\ ,
\end{aligned}$$

and

$$T_3=\frac{1}{2}(z_1\partial_1-z_2\partial_2-\overline{z}_1\overline{\partial}_1+\overline{z}_2\overline{\partial}_2)\ ,$$

$$T_8=\frac{1}{2\sqrt{3}}(z_1\partial_1+z_2\partial_2-\overline{z}_1\overline{\partial}_1-\overline{z}_2\overline{\partial}_2-2z_3\partial_3+2\overline{z}_3\overline{\partial}_3)\ .$$

These act as hermitian operators on holomorphic functions of $z_{1,2,3}$ and $\overline{z}_{1,2,3}$, normalized with respect to the inner product

$$(f,g)\equiv\int d^3z d^3\overline{z}\ e^{-\sum_i|z_i|^2}\ f^*(z,\overline{z})\ g(z,\overline{z})\ .$$

It is convenient to introduce the positive integer Dynkin labels a_1 and a_2, for which

$$T_3|\ a_1,a_2>=\frac{a_1}{2}|\ a_1,a_2>\ ,$$

and

$$T_8|\ a_1,a_2>\ =\ \frac{1}{2\sqrt{3}}(a_1+2a_2)|\ a_1,a_2>\ .$$

The highest-weight states of each $SU(3)$ representation are holomorphic polynomials of the form

$$z_1^{a_1}\overline{z}_3^{a_2}\ ,$$

where a_1,a_2 are the Dynkin indices, since it is easily seen to reproduce the above values for T_3 and T_8. This describes all representations of $SU(3)$ as homogeneous holomorphic polynomials. Finally we note that any function of the quadratic invariant

$$Z^2\equiv|z_1|^2+|z_2|^2+|z_3|^2\ ,$$

can multiply these polynomials without affecting their $SU(3)$ transformation properties.

5.5. Solutions of Kostant's Equation

Kostant's equation

$$\not{K}\Psi\ =\ \sum_{a=4,5,6,7}\gamma^aT_a\Psi\ =\ 0\ .$$

now becomes two coupled systems of equations

$$\begin{aligned}
(z_1\partial_3-\overline{z}_3\overline{\partial}_1)\psi_1+(z_2\partial_3-\overline{z}_3\overline{\partial}_2)\psi_2 &= 0\ ,\\
(z_3\partial_1-\overline{z}_1\overline{\partial}_3)\psi_2-(z_3\partial_2-\overline{z}_2\overline{\partial}_3)\psi_1 &= 0\ ,
\end{aligned}$$

and

$$\begin{aligned}
(z_3\partial_1-\overline{z}_1\overline{\partial}_3)\psi_0-(z_2\partial_3-\overline{z}_3\overline{\partial}_2)\psi_{12} &= 0\ ,\\
(z_3\partial_2-\overline{z}_2\overline{\partial}_3)\psi_0+(z_1\partial_3-\overline{z}_3\overline{\partial}_1)\psi_{12} &= 0\ .
\end{aligned}$$

The homogeneity operators

$$D=z_1\partial_1+z_2\partial_2+z_3\partial_3\ ,\qquad \overline{D}=\overline{z}_1\overline{\partial}_1+\overline{z}_2\overline{\partial}_2+\overline{z}_3\overline{\partial}_3$$

commute with $\not{K}$, allowing the solutions of the Kostant equation to be arranged as irreps of the $SU(2)\times U(1)$ generated by the operators

$$L_i=T_i+S_i\ ,\ i=1,2,3\ ;\qquad L_8=T_8+S_8\ .$$

The solutions for each triplet, conveniently written only for the highest weight states, are of the form

$$\Psi\ =\ z_3^{a_1}\ \overline{z}_2^{a_2}\ :\qquad [a_2]_{-\frac{2a_1+a_2+3}{6}}\ ,$$

$$\Psi \;=\; \theta_1\, z_1^{a_1}\, \overline{z}_2^{a_2} \;:\; \qquad [a_1+a_2+1]_{\frac{a_1-a_2}{6}} \;,$$
$$\Psi \;=\; \theta_1\theta_2\, z_1^{a_1}\, \overline{z}_3^{a_2} \;:\; \qquad [a_1]_{\frac{2a_2+a_1+3}{6}} \;, \qquad (42)$$

where we have indicated their $SU(2)$ Dynkin labels. All other solutions in the same Euler triplet can be obtained by repeated action of the lowering operator

$$L_1 - iL_2 = \theta_2 \frac{\partial}{\partial\theta_1} + (z_2\partial_1 - \overline{z}_1\overline{\partial}_2) \;.$$

We now see how the triplets arise as polynomials of the same degree.

5.6. The Poincaré Algebra

It is easy te represent the Poincaré algebra on the Euler triplets. Starting from the more general representation

$$\begin{aligned}
M^{12} &= i(x\overline{p} - \overline{x}p) + \frac{1}{2}\theta_m\frac{\partial}{\partial\theta_m} - \frac{1}{2}\overline{\theta}^m\frac{\partial}{\partial\overline{\theta}^m} + S^{12} \;, \\
M^{+-} &= -x^-p^+ - \frac{i}{2}\theta_m\frac{\partial}{\partial\theta_m} - \frac{i}{2}\overline{\theta}^m\frac{\partial}{\partial\overline{\theta}^m} \;, \\
M^{+} &= -xp^+ \;, \qquad \overline{M}^+ = -\overline{x}p^+ \;, \\
M^{-} &= x^-p - \frac{1}{2}\{x, P^-\} + i\frac{p}{p^+}(\theta_m\frac{\partial}{\partial\theta_m} + S^{12}) \;, \\
\overline{M}^{-} &= x^-\overline{p} - \frac{1}{2}\{\overline{x}, P^-\} + i\frac{\overline{p}}{p^+}(\overline{\theta}^m\frac{\partial}{\partial\overline{\theta}^m} - S^{12}) \;,
\end{aligned}$$

and identify the rest of the helicity generator as

$$S^{12} \;=\; \frac{1}{\sqrt{3}}T_8 \;. \qquad (43)$$

These act on chiral superfields whose entries are polynomials in z and $\overline{z}$.

There is no difficulty in writing a *free* Lagrangian for these solutions, but one could contemplate writing a free Lagrangian for all the solutions at one fell swoop. For this, we need to consider a chiral superfield whose entries are arbitrary polynomials in the complex variables, but subject to the Kostant equation *as a constraint.* The action would be of the form

$$S = \int d^2x dx^- \int d^3z d^3\overline{z}\; \mathcal{L} \;, \qquad (44)$$

suggesting that the classical space includes the group manifold as well. The Lagrange density would then be built out of chiral superfields of the form $\Psi(y^-, x^i, z_a, \overline{z}_a, \theta_i)$, which satisfy *both* the chiral and Kostant constraints. Work is in progress to see if one can introduce interactions among these light-cone superfields .

6. Outlook

This example only serves to introduce the method we intend to pursue. The interesting case in eleven dimensions singles out the coset $F_4/SO(9)$. A similar procedure will lead us to consider light-cone superfields over 4 copies of 26 real variables[19], and eight Grassmann variables. The light-cone Lorentz group generators are now built out of the $SO(9)$ generators, of the form

$$L^{ij} \;=\; T^{ij} + S^{ij} \;, \quad i,j = 1,2\ldots,9 \;, \qquad (45)$$

in which T^{ij} generate the $SO(9)$ little group in the particular representation associated with each Euler triplet, and they act along the 256×256 unit matrix. In particular, $T^{ij} = 0$ for the supergravity multiplet, and the "spin" part is

$$S^{ij} \;=\; -\frac{i}{4}\sum_{a,b=1}^{16} f^{ij\,ab}\,\gamma^{ab} \;. \qquad (46)$$

Here the $f^{ij\,ab}$ are the structure functions of the exceptional group F_4! In closing we note that the same coset plays a prominent role in the projective geometry associated with the Exceptional Jordan Algebra, leading one to hope in a new interpretation with $SO(9)$ as a space group. Details will be presented elsewhere.

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REFERENCES

1. G. L. Stavraki, International School of Theoretical Physics, Yalta (1966); *Physics of High Energy and Theoretical Elementary Particle Physics. Ukrain. Acad Sci./JINR*, Kiev (1967)
2. H. Miyazawa, *Phys. Rev.* **170**, 1958(1968)
3. Yu.A. Golfand and E.P. Likhtman,*JETP Lett.* **13**, 323(1971; *Pisma Zu. Eksp. Teor. Fiz.* **13** 452(1971)
4. D.V. Volkov and V. P. Akulov, *JETP Lett.* **16**, 438(1972).
5. R. Dolen, D. Horn, C. Schmid, it Phys. Rev. **166**, 1768(1968)
6. P. Ramond, *Phys. Rev* **D3**, 988(1971)
7. A. Neveu and J. H. Schwarz, *Phys. Rev.* **D4**, 1109(1971)
8. J. L. Gervais and B. Sakita, *Phys. Rev.* **D4**, 2291(1971)
9. J. Wess, B. Zumino, *Phys. Lett.* **B49**, 52(1974)
10. F. Gliozzi, J Scherk, and D. Olive *Nucl.Phys.* **B122**, 253(1977)
11. See, W. G. McKay, J. Patera, Tables of Dimensions, Indices, and Branching Rules for Representations of Simple Lie Algebras, Marcel Dekker (New York and Basel), 1981.
12. B. Gross, B. Kostant, P. Ramond, and S. Sternberg, *Proc. Nat'l Acad. Sci. USA* **95**, 8441(1998).
13. B. Kostant, *Duke J. Math.* **100**, 447(1999)
14. Loyal Durand III, *Phys. Rev.* **128**, 434(1962); K. Case and S. Gasiorowicz, *Phys. Rev* **125**, 1055(1962)
15. E. Witten and S. Weinberg, *Phys. Lett.* **B96**, 59(1980)
16. C. Aragone and S. Deser, *Nuovo Cim.* **57B**, 33(1980)
17. Anders K.H. Bengtsson, Ingemar Bengtsson, Lars Brink, *Nucl. Phys.* **B227**, 31(1983)
18. This work has been done in collaboration with L. Brink; see also Lars Brink, Pierre Ramond, *Dirac Equations, Light Cone Supersymmetry, and Superconformal Algebras.* In *Shifman, M.A. (ed.): The many faces of the superworld* 398-416. [hep-th 9908208], and references therein.
19. T. Fulton, *J. Phys. A: Math. Gen.* **18**, 2863(1985)